Journal of Sound and Vibration (2001) **246**(1), 47–62
doi:10.1006/jsvi.2000.3627, available online at http://www.idealibrary.com on IDEAL®

VIBRATION AND BUCKLING OF MULTILAYERED COMPOSITE BEAMS ACCORDING TO HIGHER ORDER DEFORMATION THEORIES

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(*Received* 3 *August* 2000, *and in final form* 28 *November* 2000)

Natural frequencies and buckling stresses of laminated composite beams are analyzed by taking into account the complete effects of transverse shear and normal stresses and rotatory inertia. By using the method of power series expansion of displacement components, a set of fundamental dynamic equations of a one-dimensional higher order theory for laminated composite beams subjected to axial stress is derived through Hamilton's principle. Several sets of truncated approximate theories are applied to solve the eigenvalue problems of a simply supported laminated composite beam. In order to ensure the accuracy of the present theory, convergence properties of the first seven natural frequencies are examined in detail. Numerical results are compared with those of the published existing theories and FEM solutions. The modal displacement and stress distributions in the depth direction are obtained and plotted in figures. The present global higher order approximate theories can predict the natural frequencies, buckling stresses and interlaminar stresses of multilayered composite beams as accurately as three-dimensional elasticity solutions.

1. INTRODUCTION

The application of composite materials to structural members has increased due to high strength/stiffness for light weight and facility to materialize fibre orientation, material and stacking sequence. However, these materials pose new problems, such as effects of transverse shear deformation due to the low ratio of transverse shear modulus to axial modulus, failure due to delamination and other secondary effects in the material formation.

The classical laminated beam theory, based on the Euler–Bernoulli hypothesis, is inaccurate for a moderately deep laminated beam with relatively soft transverse shear modulus and for highly anisotropic composites. The inaccuracy is due to neglecting the transverse shear and normal strains in the laminate. In order to take into account the effects of low ratio of transverse shear modulus to the in-plane modulus, the first order shear deformation theory of Timoshenko has been developed. However, since in the theory the transverse shear strain is assumed to be constant in the depth direction, a shear correction factor has to be incorporated to adjust the transverse shear stiffness for studying the static or dynamic problems of beams. The accuracy of solutions of the first order shear deformation theory will be strongly dependent on predicting better estimates for the shear correction factor κ^2. It has been shown that the classical and first order shear deformation theories are inadequate to predict the accurate solutions of laminated composite beams.

In order to obtain accurate predictions of the stress distributions of simply supported laminated composite plates and beams subjected to sinusoidal loading, Pagano [1, 2] has developed the three-dimensional elasticity theory. However, dynamic problems of laminated composite beams have not been studied as extensively as plates. Accurate solutions based on the three-dimensional elasticity theory are often computationally expensive.

Several approximately refined one-dimensional higher order theories have been proposed to analyze the response characteristics of beams. In recent years discrete layerwise theories have been presented to obtain more accurate information on the ply level. Based on a generalized laminate plate theory, a finite-element model of laminated composite plates that accurately describes the three-dimensional effects has been developed by Robbins and Reddy [3]. Although such theories have been proved to be one of the best alternatives to three-dimensional elasticity theories, these theories require numerous unknowns for multilayer laminates and are often computationally expensive in obtaining accurate solutions. Since the total number of unknowns is dependent on the number of layers in a laminate, the number of unknowns will increase dramatically as the number of layers increases. Aitharaju and Averill [4] have developed a finite-element formulation for the stress analysis of laminated composite beams subjected to sinusoidal loading. The theory assumes a zig-zag distribution of the in-plane displacement field with satisfying transverse shear continuity at the lamina interfaces and the transverse normal stress to be constant through the thickness of the laminate. Since the stiffness matrix of the element can be stacked in the depth direction, the number of degrees of freedom is independent of the number of layers.

A number of single-layer (global) higher order beam theories that include the effects of transverse shear deformations have been published in the literature. Although various models of higher order displacement fields have been considered, most of these theories are the third order theories in which the axial displacements are assumed to be a cubic expression of the depth co-ordinate and the transverse displacement to be a quadratic expression at most. It has been pointed out that, in general, single-layer second and third order theories are adequate in representing global responses, such as natural frequencies and buckling stresses, but inadequate in representing the local responses, such as stress distributions in each layer of laminates. It seems to be a hasty conclusion to analyze the response characteristics of laminated composite beams by using single-layer higher order theories. For a deep isotropic beam, a one-dimensional higher order theory has been developed and has been applied to the statics and dynamics of a very deep beam by Matsunaga [5–7]. Natural frequencies and buckling stresses of deep isotropic beams subjected to axial stress have been analyzed by using the approximate one-dimensional higher order theories. Remarkable effects of transverse shear deformation and depth change have been predicted in the results. For the vibration and stability problems of general cross-ply composite plates, a global higher order theory has been developed by Matsunaga [8]. It has been shown that a global higher order plate theory can predict accurate results not only for the natural frequency and buckling stress but also for the distribution of displacements and stress components in cross-ply multilayered composite plates. However, general higher order theories of beams which take into account the complete effects of shear and normal strains and rotatory inertia have not been investigated in the vibration and stability problems of multilayered composite beams.

This paper presents a global higher order theory for analyzing natural frequencies and buckling stresses of laminated composite beams. The complete effects of transverse shear and normal stresses and rotatory inertia can be taken into account within the approximate one-dimensional theory. Several sets of the governing equations of truncated approximate theories are applied to the analysis of vibration and buckling problems of a simply supported multilayered elastic beam subjected to axial stress. Based on the power series expansions of continuous displacement components, a fundamental set of equations of a one-dimensional higher order beam theory is derived through Hamilton's principle. Natural frequencies and buckling stresses of a laminated composite beam subjected to axial stress are obtained by solving the eigenvalue problem numerically. Convergence properties of the present numerical solutions are shown to be accurate for the natural frequencies with respect to the order of approximate theories. Several comparisons of the present results are also made with previously published results. For multilayered composite beams the distribution of modal displacements and modal stresses in the depth direction has also been obtained accurately in the ply level. The modal transverse stresses have been obtained by integrating the three-dimensional equations of motion in the depth direction starting from the top or bottom surface of the laminated beams. The total number of unknowns depend on the order of approximate theories, but is not dependent on the number of layers in any multilayered beams. The present results obtained by various sets of approximate theories are considered to be accurate enough for general laminated composite beams with small length-to-depth ratio. One-dimensional global higher order theory in the present paper can predict the natural frequencies, buckling stresses and modal stress distributions of simply supported multilayered composite beams accurately when compared to the published solutions in the literature.

2. FUNDAMENTAL EQUATIONS OF LAMINATED COMPOSITE BEAMS

Consider a straight uniform laminated beam of length L as shown in Figure 1, having a rectangular cross-section of depth H which consists of K-layers and width B which is assumed to be sufficiently small relative to the depth. A Cartesian co-ordinate system (x, y, z) is defined on the central axis of the beam, where the x-axis is taken along this axis with the y-axis in the width direction and the z-axis in the depth direction. Assuming that the deformations of the beam take place in the x–z plane, the displacement components in a beam can be expressed as

$$u \equiv u(x, z; t), \qquad v \equiv v(x, z; t) = 0, \qquad w \equiv w(x, z; t), \tag{1}$$

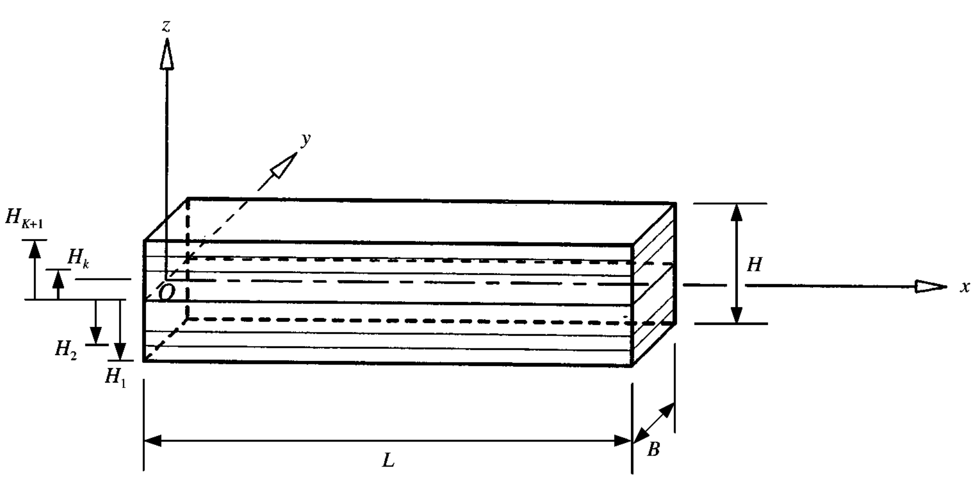

Figure 1. K-layers cross-ply laminated composite beam and co-ordinates.

where t denotes time. The displacement components may be expanded into power series of the depth co-ordinate z as follows:

$$u = \sum_{n=0}^{\infty} \overset{(n)}{u} z^n, \qquad w = \sum_{n=0}^{\infty} \overset{(n)}{w} z^n, \tag{2}$$

where $n = 0, 1, 2, \ldots, \infty$.

Based on this expression of the displacement components, a set of the linear fundamental equations of a one-dimensional higher order theory of beams can be summarized in the following.

2.1. STRAIN–DISPLACEMENT RELATIONS

Strain components may also be expanded as follows:

$$\varepsilon_{xx} = \sum_{n=0}^{\infty} \overset{(n)}{\varepsilon}_{xx} z^n, \qquad \gamma_{xz} = \gamma_{zx} = \sum_{n=0}^{\infty} \overset{(n)}{\gamma}_{xz} z^n, \qquad \varepsilon_{zz} = \sum_{n=0}^{\infty} \overset{(n)}{\varepsilon}_{zz} z^n \tag{3}$$

and strain–displacement relations can be written as

$$\overset{(n)}{\varepsilon}_{xx} = \overset{(n)}{u}_{,x}, \qquad \overset{(n)}{\gamma}_{xz} = \overset{(n)}{\gamma}_{zx} = \tfrac{1}{2}\left\{(n+1)\overset{(n+1)}{u} + \overset{(n)}{w}_{,x}\right\}, \qquad \overset{(n)}{\varepsilon}_{zz} = (n+1)\overset{(n+1)}{w}, \tag{4}$$

where a comma denotes partial differentiation with respect to the co-ordinate subscripts that follow.

2.2. EQUATIONS OF MOTION AND BOUNDARY CONDITIONS

Under the assumption of plane strain or plane stress in the width direction, by introducing stress components σ_{xx}, $\tau_{xz} = \tau_{zx}$ and σ_{zz}, Hamilton's principle is applied to derive the equations of motion and natural boundary conditions of a beam. In order to treat vibration and stability problems of a beam subjected to uniformly distributed axial stress $\sigma_0 = \sigma_0(z)$, additional work due to this stress which is assumed to remain unchanged during vibrating and/or buckling is taken into consideration. It is also assumed that stresses are free on the top and bottom surfaces of the beam.

The principle for the present problems may be expressed as follows:

$$\int_{t_1}^{t_2}\int_V \{\sigma_{xx}\delta\varepsilon_{xx} + 2\tau_{xz}\delta\gamma_{xz} + \sigma_{zz}\delta\varepsilon_{zz} + \sigma_0(u_{,x}\delta u_{,x} + w_{,x}\delta w_{,x}) - \rho(\dot{u}\delta\dot{u} + \dot{w}\delta\dot{w})\}\,\mathrm{d}V\,\mathrm{d}t = 0, \tag{5}$$

where the overdot indicates partial differentiation with respect to time and $\rho = \rho(z)$ denotes the mass density, $\mathrm{d}V$, the volume element, σ_0, the axial stress which is assumed to be expanded as follows:

$$\sigma_0 = \sum_{\ell=0}^{\infty} \sigma_0^{(\ell)} z^{\ell}, \tag{6}$$

where $\ell = 0, 1, 2, \ldots, \infty$.

By performing the integration over the area of cross-section of the beam and the variation as indicated in equation (5), the equations of motion are obtained as follows:

$$\delta \overset{(n)}{u}: \quad \overset{(n)}{N}_{,x} - n \overset{(n-1)}{Q} + \sum_{m=0}^{\infty} \left[\sum_{\ell=0}^{\infty} \overset{(n+m+\ell+1)}{\sigma_0} \overset{(m)}{u}_{,xx} - \overset{(n+m+1)}{\rho} \overset{(m)}{\ddot{u}} \right] = 0, \tag{7}$$

$$\delta \overset{(n)}{w}: \quad \overset{(n)}{Q}_{,x} - n \overset{(n-1)}{T} + \sum_{m=0}^{\infty} \left[\sum_{\ell=0}^{\infty} \overset{(n+m+\ell+1)}{\sigma_0} \overset{(m)}{w}_{,xx} - \overset{(n+m+1)}{\rho} \overset{(m)}{\ddot{w}} \right] = 0, \tag{8}$$

where $n, m = 0, 1, 2, \ldots, \infty$ and

$$\overset{(n+m+\ell+1)}{\sigma_0} = \sum_{k=1}^{K} \sigma_0^{(k)} \frac{H_{k+1}^{n+m+\ell+1} - H_k^{n+m+\ell+1}}{n+m+\ell+1}, \qquad \overset{(n+m+1)}{\rho} = \sum_{k=1}^{K} \rho^{(k)} \frac{H_{k+1}^{n+m+1} - H_k^{n+m+1}}{n+m+1}, \tag{9}$$

with $\sigma_0^{(k)}$, $\rho^{(k)}$ and H_k denoting the initial axial stress, mass density of kth layer and thickness co-ordinate of the lower side of kth layer, respectively, and K denotes the total number of layers in the laminated beams.

The stress resultants are defined as follows:

$$\overset{(n)}{N} = \sum_{k=1}^{K} \sigma_{xx}^{(k)} \frac{H_{k+1}^{n+1} - H_k^{n+1}}{n+1}, \quad \overset{(n)}{Q} = \sum_{k=1}^{K} \tau_{xz}^{(k)} \frac{H_{k+1}^{n+1} - H_k^{n+1}}{n+1}, \quad \overset{(n)}{T} = \sum_{k=1}^{K} \sigma_{zz}^{(k)} \frac{H_{k+1}^{n+1} - H_k^{n+1}}{n+1}. \tag{10}$$

For the boundary conditions at the ends on the central axis, the quantities

$$\overset{(n)}{u} \quad \text{or} \quad \overset{(n)}{N} + \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \overset{(n+m+\ell+1)}{\sigma_0} \overset{(m)}{u}_{,x} \tag{11}$$

$$\overset{(n)}{w} \quad \text{or} \quad \overset{(n)}{Q} + \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \overset{(n+m+\ell+1)}{\sigma_0} \overset{(m)}{w}_{,x} \tag{12}$$

are to be prescribed.

2.3. CONSTITUTIVE RELATIONS

For elastic and orthotropic materials of each layer of laminated composite beams, the two-dimensional constitutive relations for the kth layer can be written under the assumption of plane stress in the width direction of the beam as

$$\begin{Bmatrix} \sigma_{xx}^{(k)} \\ \sigma_{zz}^{(k)} \\ \tau_{xz}^{(k)} \end{Bmatrix} = \begin{Bmatrix} C_{11}^{(k)} & C_{12}^{(k)} & 0 \\ C_{21}^{(k)} & C_{22}^{(k)} & 0 \\ 0 & 0 & C_{33}^{(k)} \end{Bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(k)} \\ \varepsilon_{zz}^{(k)} \\ \gamma_{xz}^{(k)} \end{Bmatrix}, \tag{13}$$

where

$$C_{11}^{(k)} = \frac{E_x^{(k)}}{1 - v_{xz}^{(k)} v_{zx}^{(k)}}, \qquad C_{12}^{(k)} = \frac{E_x^{(k)} v_{zx}^{(k)}}{1 - v_{xz}^{(k)} v_{zx}^{(k)}},$$

$$C_{21}^{(k)} = \frac{E_z^{(k)}\, \nu_{xz}^{(k)}}{1 - \nu_{xz}^{(k)}\, \nu_{zx}^{(k)}}, \qquad C_{22}^{(k)} = \frac{E_z^{(k)}}{1 - \nu_{xz}^{(k)}\, \nu_{zx}^{(k)}},$$

$$C_{33}^{(k)} = 2G_{xz}^{(k)}, \tag{14}$$

with $E_x^{(k)}$ and $E_z^{(k)}$ being Young's modulus and $\nu_{xz}^{(k)}$ and $\nu_{zx}^{(k)}$ are the Poisson ratio. The first suffix of ν denotes the direction of stress and the second, that of expansion and contraction. The following relation can be established by the reciprocal theorem:

$$E_x \nu_{zx} = E_z \nu_{xz}. \tag{15}$$

2.4. STRESS RESULTANTS IN TERMS OF THE EXPANDED DISPLACEMENT COMPONENTS

Stress resultants can be derived from equations (10) and (13) in terms of the expanded displacement components as follows:

$$\overset{(n)}{N} = \sum_{m=0}^{\infty} [\overset{(n+m+1)}{C_{11}} \overset{(m)}{\varepsilon}_{xx} + \overset{(n+m+1)}{C_{12}} \overset{(m)}{\varepsilon}_{zz}] = \sum_{m=0}^{\infty} [\overset{(n+m+1)}{C_{11}} \overset{(m)}{u}_{,x} + (m+1) \overset{(n+m+1)}{C_{12}} \overset{(m+1)}{w}],$$

$$\overset{(n)}{T} = \sum_{m=0}^{\infty} [\overset{(n+m+1)}{C_{21}} \overset{(m)}{\varepsilon}_{xx} + \overset{(n+m+1)}{C_{22}} \overset{(m)}{\varepsilon}_{zz}] = \sum_{m=0}^{\infty} [\overset{(n+m+1)}{C_{21}} \overset{(m)}{u}_{,x} + (m+1) \overset{(n+m+1)}{C_{22}} \overset{(m+1)}{w}],$$

$$\overset{(n)}{Q} = \sum_{m=0}^{\infty} \overset{(n+m+1)}{C_{33}} \overset{(m)}{\gamma}_{xz} = \sum_{m=0}^{\infty} \frac{1}{2} \overset{(n+m+1)}{C_{33}} [(m+1) \overset{(m+1)}{u} + \overset{(m)}{w}_{,x}\}, \tag{16}$$

where

$$\overset{(n+m+1)}{C_{ij}} = \sum_{k=1}^{K} C_{ij}^{(k)} \frac{H_{k+1}^{n+m+1} - H_k^{n+m+1}}{n+m+1}, \quad (i,\ j = 1, 2, 3). \tag{17}$$

2.5. EQUATIONS OF MOTION IN TERMS OF THE EXPANDED DISPLACEMENT COMPONENTS

The equations of motion can be expressed in terms of the expanded displacement components by using equations (16) as

$$\delta \overset{(n)}{u}: \quad \sum_{m=0}^{\infty} \{[\overset{(n+m+1)}{C_{11}} \overset{(m)}{u}_{,x} + (m+1) \overset{(n+m+1)}{C_{12}} \overset{(m+1)}{w}]_{,x} - \overset{(n+m+1)}{\sigma} \overset{(m)}{\ddot{u}}$$

$$- \frac{n}{2} \overset{(n+m)}{C_{33}} [(m+1) \overset{(m+1)}{u} + \overset{(m)}{w}_{,x}] + \sum_{\ell=0}^{\infty} \overset{(n+m+\ell+1)}{\sigma_0} \overset{(m)}{u}_{,xx}\} = 0, \tag{18}$$

$$\delta \overset{(n)}{w}: \quad \sum_{m=0}^{\infty} \{\tfrac{1}{2} \overset{(n+m+1)}{C_{33}} [(m+1) \overset{(m+1)}{u} + \overset{(m)}{w}_{,x}]_{,x} - \overset{(n+m+1)}{\rho} \overset{(m)}{\ddot{w}} - n [\overset{(n+m)}{C_{21}} \overset{(m)}{u}_{,x}$$

$$+ \overset{(n+m)}{C_{22}} (m+1) \overset{(m+1)}{w} + \sum_{\ell=0}^{\infty} \overset{(n+m+\ell+1)}{\sigma_0} \overset{(m)}{w}_{,xx}\} = 0. \tag{19}$$

2.6. *M*TH ORDER APPROXIMATE THEORY

Since the fundamental equations mentioned above are complex, approximate theories of various orders may be considered for the present problem. A similar set of the following combination of Mth ($M \geqslant 1$) order approximate equations is proposed in the case of isotropic beams by Matsunaga [5–7]. This combination of the selected terms of displacement components is suggested from the form of shear strain components in equation (4) as follows:

$$u = \sum_{m=0}^{2M-1} \overset{(m)}{u} z^m, \qquad w = \sum_{m=0}^{2M-2} \overset{(m)}{w} z^m, \tag{20}$$

where $m = 0, 1, 2, 3, \ldots$. The total number of the unknown displacement components is $(4M - 1)$.

In the above cases of $M = 1$, an assumption that the normal strain ε_{zz} is zero is inherently imposed. Another set of the governing equations of the lowest order approximate theory ($M = 1^*$) is derived with the use of an assumption that the normal stress σ_{zz} is zero. For flexural problems, this theory corresponds to the Timoshenko beam theory with the shear correction coefficient $\kappa^2 = 1$. Under this assumption, if the shear strain γ_{xz} vanishes through the depth of a beam, the lowest order approximate theory reduces to the classical beam theory.

3. NAVIER SOLUTION FOR SIMPLY SUPPORTED BEAMS

In order to show the applicability and reliability of the present one-dimensional higher order theories for the analysis of vibration and buckling problems of a laminated composite beam, a simply supported beam (with movable hinged end) subjected to axial stress is analyzed. In the following analysis, the axial stress σ_0 is assumed to distribute uniformly in the depth direction. Only the first term of the expanded in-plane stress (6) is considered, i.e., $\sigma_0 = \sigma_0^{(0)}$ which is the same for each layer.

Boundary conditions (11) and (12) can be expressed at the x-constant ends

$$\overset{(n)}{u}_{,x} = 0, \qquad \overset{(n)}{w} = 0. \tag{21}$$

Since a beam is in a state of uniform stresses, the axial stress is considered to be constant during vibrating and/or buckling. Following the Navier solution procedure, displacement components that satisfy the equations of boundary conditions (21) may be expressed as

$$\overset{(n)}{u} = \sum_{r=1}^{\infty} \overset{(n)}{u}_r \cos\frac{r\pi x}{L}\, e^{i\omega t}, \quad \overset{(n)}{w} = \sum_{r=1}^{\infty} \overset{(n)}{w}_r \sin\frac{r\pi x}{L}\, e^{i\omega t}, \tag{22}$$

where the displacement mode number $r = 1, 2, 3, \ldots, \infty$.

The equations of motion are rewritten in terms of the generalized displacement components $\overset{(n)}{u}_r$ and $\overset{(n)}{w}_r$.

The dimensionless axial or buckling stress Λ is defined as follows:

$$\Lambda = \frac{AL^2}{\pi^2 I} \frac{\sigma_0}{E_x^{(1)}}, \tag{23}$$

where

$$A = BH, \qquad I = BH^3/12. \tag{24}$$

The dimensionless frequency Ω is defined as follows:

$$\Omega = \omega H \sqrt{\rho^{(1)}/E_x^{(1)}}. \tag{25}$$

4. EIGENVALUE PROBLEM FOR VIBRATION AND BUCKLING PROBLEMS

The equations of motion (18) and (19) can be rewritten by collecting the coefficients for the generalized displacements of any fixed value r. The generalized displacement vector $\{\mathbf{U}\}$ is expressed as

$$\{\mathbf{U}\}^{\mathrm{T}} = \{ \overset{(0)}{u}_r, \ldots, \overset{(2M-1)}{u_r}; \qquad \overset{(0)}{w}_r, \ldots, \overset{(2M-2)}{w_r} \}. \tag{26}$$

The dynamic equation can be expressed as the following eigenvalue problem:

$$([\mathbf{K}] - \Omega^2 [\mathbf{M}]) \{\mathbf{U}\} = \{\mathbf{0}\}, \tag{27}$$

where the matrix $[\mathbf{K}]$ denotes the stiffness matrix which may contain the terms of the axial stress and matrix $[\mathbf{M}]$, the mass matrix.

For buckling problems, the natural frequency vanishes and the stability equation can be expressed as the following eigenvalue problem:

$$([\mathbf{K}] + \Lambda [\mathbf{S}]) \{\mathbf{U}\} = \{\mathbf{0}\}, \tag{28}$$

where the matrix $[\mathbf{K}]$ denotes the stiffness matrix and matrix $[\mathbf{S}]$, the geometric-stiffness matrix due to the axial stress.

The number of eigenvalues is the same as that of the components of the generalized displacement vector for each displacement mode number of r. Although all the eigenvalues and eigenvectors can be computed, the dominant eigenvalue which corresponds to the minimum natural frequency or the critical buckling stress is of great concern. When the lowest natural frequency vanishes, the axial stress reduces to the critical buckling stress of the beam.

5. DETERMINATION OF MODAL STRESS DISTRIBUTION

Although the transverse stress components can be calculated from the constitutive relations, these stresses may not satisfy the continuity conditions at the interface between layers and stress boundary conditions on the top and bottom surfaces of a laminated beam. Axial stress component has no reference to the surface boundary conditions and can be obtained by the constitutive relations. The transverse shear and normal stresses are determined by integrating the equations of motion of a three-dimensional elastic continuum. The modal axial stress of the kth layer can be derived in terms of the expanded displacement components by introducing the strain–displacement relations (4) into the constitutive relations (13). The modal transverse stresses of the kth layer are obtained by integrating the three-dimensional equations of motion in the depth direction starting from the top (or bottom) surface of the laminated beam as follows:

$$\tau_{xz}^{(k)} = -\int_{H_k}^{z} (\sigma_{xx,x} - \rho \ddot{u})\, \mathrm{d}z + C_1^{(k)}(x), \quad \sigma_{zz}^{(k)} = -\int_{H_k}^{z} (\tau_{xz,x} - \rho \ddot{w})\, \mathrm{d}z + C_2^{(k)}(x), \tag{29}$$

where $C_1^{(k)}$ and $C_2^{(k)}$ are constants obtained from the stress conditions on the top and bottom surfaces of the kth layer. If the boundary conditions of transverse stresses are prescribed on one of the top or bottom surfaces, the stress boundary conditions on the other surface can be satisfied through the equations of motion (18) and (19). Because of the discontinuity of the axial stress at layer interfaces, the integration is performed in a piecewise manner. The modal stress components in the kth layer of laminated composite beams can be expressed as follows:

$$\sigma_{xx}^{(k)} = \sum_{n=0}^{\infty} [F_1]^{(k)} z^n, \tag{30}$$

$$\tau_{xz}^{(k)} = -\sum_{n=0}^{\infty} [F_{1,x} + \rho\omega^2 \overset{(n)}{u}]^{(k)} \frac{[z^{n+1} - H_k^{n+1}]}{n+1} + \tau_{xz}^{(k-1)}\Big|_{z=H_k}, \tag{31}$$

$$\sigma\tau_{zz}^{(k)} = \sum_{n=0}^{\infty} [F_{1,xx} + \rho\omega^2 \overset{(n)}{u}_{,x}]^{(k)} \left[\frac{z^{n+2} - H_k^{n+2}}{(n+1)(n+2)} - \frac{H_k^{n+1}(z - H_k)}{n+1}\right]$$
$$- \rho\omega^2 \sum_{n=0}^{\infty} w^{(n)} \frac{[z^{n+1} - H_k^{n+1}]}{n+1} - \tau_{xz,x}^{(k-1)}|_{z=H_k} [z - H_k] + \sigma_{zz}^{(k-1)}|_{z=H_k}, \tag{32}$$

where

$$F_1 \equiv C_{11} \overset{(n)}{u}_{,x} + (n+1) C_{12} \overset{(n+1)}{w}. \tag{33}$$

6. NUMERICAL EXAMPLES AND RESULTS

6.1. NUMERICAL EXAMPLES

The effects of transverse shear and normal stresses and rotatory inertia on natural frequencies and buckling stresses of a simply supported cross-ply laminated composite beam subjected to axial stresses are studied through numerical examples. The following sets of orthotropic material constants of each layer are taken to be the same in all the layers and the fibre orientations may be different among layers.

Material 1 [9, 10]:

$$E_L^{(k)} = 14{\cdot}5 \times 10^{10}\ \text{N/m}^2, \quad E_T^{(k)} = 0{\cdot}96 \times 10^{10}\ \text{N/m}^2, \quad G_{LT}^{(k)} = 0{\cdot}41 \times 10^{10}\ \text{N/m}^2,$$
$$G_{TT}^{(k)} = 0{\cdot}34 \times 10^{10}\ \text{N/m}^2, \qquad \nu_{LT} = \nu_{TT} = 0{\cdot}3. \tag{34}$$

Material 2 [11]:

$$E_L^{(k)} = 40 E_T^{(k)}, \quad G_{LT}^{(k)} = 0{\cdot}6\, E_T, \quad G_{TT}^{(k)} = 0{\cdot}5\, E_T, \quad \nu_{TL} = \nu_{TT} = 0{\cdot}25. \tag{35}$$

Material 3 [12–14]:

$$E_L = 144{\cdot}8\ \text{GPa}, \quad E_T = 9{\cdot}65\ \text{GPa}, \quad G_{LT} = 4{\cdot}14\ \text{GPa}, \quad G_{TT} = 3{\cdot}45\ \text{GPa}, \quad \nu_{LT} = 0{\cdot}3. \tag{36}$$

Material 4 [1, 2]:

$$E_L^{(k)} = 25 \times 10^6\ \text{psi}, \quad E_T^{(k)} = 10^6\ \text{psi}, \quad G_{LT}^{(k)} = 0{\cdot}5 \times 10^6\ \text{psi},$$
$$G_{TT}^{(k)} = 0{\cdot}2 \times 10^6\ \text{psi}, \qquad \nu_{LT} = \nu_{TT} = 0{\cdot}25. \tag{37}$$

The lower suffixes L and T signify the direction parallel to the fibres and the transverse direction respectively. The fibre orientations of the different laminas alternate between 0° and 90° with respect to the x-axis. The thickness of each layer is identified for the 0° and 90° layers in the laminates. Both symmetric and non-symmetric laminations with respect to the middle plane are considered. The axial stress σ_0 is identical in each layer. The mass density is also assumed to be uniform in the thickness direction, i.e., $\rho^{(1)}, \rho^{(2)}, \ldots, \rho^{(k)}$ are identical.

In the symmetrical laminates having odd number of layers, the 0° layers are at the outer surfaces of the laminate. For the symmetric cases, since the longitudinal and transverse vibrations are not coupled, the mechanical responses of lateral vibrations become identical both for movable and immovable hinged ends. While, for the non-symmetric cases, all modes are coupled axially and laterally, the mechanical responses of lateral vibrations become different between movable and immovable ends.

All the numerical results are obtained for the case of plane stress in the width direction and are shown in the dimensionless quantities. The absolute values of buckling stresses are shown in the following results.

6.2. CONVERGENCE OF NATURAL FREQUENCIES AND COMPARISON WITH THOSE OF EXISTING SOLUTIONS

In order to verify the accuracy of the present solutions, convergences of the first seven natural frequencies of cross-ply laminated composite beams without axial stress of the present approximate theories are examined in Table 1. Comparisons with existing results [9, 10] are also performed for simply supported symmetric [0°/90°/90°/0°] cross-ply laminated composite beams of Material 1 with the depth parameter $L/H = 10$ as also shown in Table 1. Using a third order shear deformation theory which is somewhat different from that of Khdeir and Reddy [11], a cross-ply symmetric laminated composite beams has been analyzed by Singh and Abdelnaser [9]. The first order shear deformation Timoshenko-type theory has been applied to obtain the natural frequencies of symmetric and non-symmetric laminated composite beams by Abramovich and Livshits [10]. Although the simply supported boundary conditions used in references [9, 10] are

TABLE 1

Comparison and convergence of frequencies of cross-ply laminated beams ([0°/90°/90°/0°]; $\Omega \times \sqrt{12}(L/H)^2$, $L/H = 10$, $r = 1$–7; *Material* 1)

Mode no.	Reference [9]†	Reference [10]‡	Present solution§					
			$M = 1$	$M = 1^*$	$M = 2$	$M = 3$	$M = 4$	$M = 5$
1	2·3189	2·3194	2·3663	2·0014	2·3101	2·3099	2·3095	2·3093
2	7·0171	7·0029	7·3350	6·5786	6·9821	6·9786	6·9771	6·9756
3	12·132	12·037	12·8043	11·9471	12·0515	12·0381	12·0366	12·0339
4	17·301	17·015	18·2488	15·3705^1	17·1373	17·1047	17·1042	17·1006
5	22·533	21·907	23·0725^1	17·4271	22·2173	22·1548	22·1535	22·1492
6	—	23·337^1	23·6063	22·8574	22·7131^1	22·5437^1	22·5321^1	22·5320^1
7	27·881	26·736	28·8910	28·2174	27·3055	27·2028	27·1965	27·1914

† Third order shear deformation theory (immovable hinged ends).
‡ First order shear deformation theory ($\kappa^2 = 5/6$, immovable hinged ends).
§ Present solutions: $M = 1$–5; $M = 1^*$ (First order shear deformation theory, $\kappa^2 = 1$).

TABLE 2

Comparison of frequencies of cross-ply laminated beams ($\Omega \times (L/H)^2$, $L/H = 10$, $r = 1$–7; *Material* 1)

Mode no.	[0°/0°]		[90°/90°]		[90°/0°]	
	Reference [10]†	$M = 5$	Reference [10]	$M = 5$	Reference [10]	$M = 5$
1	8·4827	8·4853	9·6974	9·7018	8·1439	4·4646
2	25·460	25·5489	36·958	37·0206	21·661	15·7873
3	43·621	44·0287	77·597	77·8567	43·788	30·5898
4	61·551	62·5462	$108{\cdot}83^1$	$108{\cdot}7873^1$	63·787	46·8575
5	79·175	80·9823	127·3	127·9560	89·150	63·7345
6	96·572	99·3602	182·75	184·0237	$89{\cdot}313^1$	$74{\cdot}3689^1$
7	$108{\cdot}83^1$	$108{\cdot}7806^1$	$217{\cdot}66^2$	$217{\cdot}3211^2$	114·30	80·8742

† First order shear deformation theory ($\kappa^2 = 5/6$, immovable hinged ends).

immovable hinged ends, the results for symmetrical laminated composite beams coincide with those for movable hinged ends. The superscript on the right shoulder of natural frequency in Table 1 is longitudial vibration mode number, r.

It is noticed that the proper order of the present higher order approximate theories may be estimated according to the level of L/H. Since the present results for $M = 1$–4 converge accurately enough within the present order of approximate theories, only the numerical results for $M = 5$ are discussed in the following.

For three types of two symmetric laminates [0°/0°] and [90°/90°] and a non-symmetric one [0°/90°], the first seven natural frequencies are compared with the results [10] in Table 2. The simply supported boundary conditions used in reference [10] are immovable hinged ends. A good agreement is obtained with the reference for the symmetric cases, while a considerable difference can be noticed for the non-symmetric case. It is noted that the simply supported boundary conditions used in reference [10] are immovable hinged ends, while the results for the present analysis are for movable hinged ends. The verification for this non-symmetric case with movable hinged ends is shown for the different numerical examples in Table 3.

Table 3 shows a comparison of the fundamental natural frequencies with the results [11] for simply supported symmetric and non-symmetric cross-ply laminated beams with movable hinged ends. A good agreement it obtained with the references both for the symmetric and non-symmetric cases. The results of reference [11] are obtained from the classical beam theory (CBT) and the first (FOBT), second (SOBT) and third order (TOBT) theories. In the third order theory, Reddy assumes parabolic distribution of the transverse shear stress that is zero at the top and bottom surfaces of the beam and neglects the transverse normal strain.

In Table 4 the first three natural frequencies of four-layers-simply supported symmetric cross-ply laminated composite beams are compared with the results in the literature [12–14]. For symmetric cross-ply laminated composite beams, free vibration frequencies based on the Timoshenko-type theory have been obtained for some boundary conditions by Chandrasekhara *et al.* [12]. Dynamic stiffness analysis of symmetric and non-symmetric cross-ply laminated beams has been presented by using a first order shear deformation theory in reference [13]. The out-of-plane free vibration problem of symmetric cross-ply

TABLE 3

Comparison of frequencies of cross-ply laminated beams ($\bar{\Omega} = \omega H \sqrt{\rho^{(1)}/E_T}(L/H)^2$, $r = 1$; *Material* 2)

Solutions	[0°/90°]		[0°/90°/0°]		[0°/90°/ ⋯] 10 layers	
	$L/H = 5$	$L/H = 10$	$L/H = 5$	$L/H = 10$	$L/H = 5$	$L/H = 10$
$M = 5$	5·6623	6·7561	9·2001	13·6079	8·0738	10·8485
$M = 4$	5·6651	6·7573	9·2001	13·6080	8·0813	10·8526
$M = 3$	5·6731	6·7615	9·2019	13·6086	8·0887	10·8560
$M = 2$	5·8335	6·8332	9·2033	13·6098	8·0921	10·8568
$M = 1$*	5·9984	6·7866	9·8078	14·1199	8·5427	11·0978
$M = 1$	6·1441	6·9910	9·8178	14·1494	8·5684	11·1533
TOBT	6·128	6·945	9·208	13·614	8·156	10·893
SOBT†	5·863	6·847	9·817	14·149	8·522	11·119
SOBT‡	5·685	6·772	9·205	13·670	8·096	10·870
FOBT‡	5·953	6·882	9·205	13·670	8·139	10·900
CBT	7·124	7·269	17·421	17·632	12·524	12·680

Note: Present solutions: $M = 2$–5; $M = 1$* (FOBT† plane stress); $M = 1$ (FOBT† plane strain). CBT, FOBT, SOBT and TOBT are cited from [11]: Classical first, second and third order beam theories († $\kappa^2 = 1$, ‡ $\kappa^2 = 5/6$).

TABLE 4

Comparison of frequencies of cross-ply laminated beams ([0°/90°/90°/0°], $\Omega \times (L/H)^2$, $L/H = 15$, $r = 1$–3; *Material* 3)

Mode no.	Reference [12]†	Reference [13]‡	Reference [14]§	$M = 5$¶
1	2·5023	2·5024	2·4959	2·4953
2	8·4812	8·4813	8·4663	8·4657
3	15·7558	15·7559	15·7384	15·7599

† Timoshenko-type theory ($\kappa^2 = 5/6$).
‡ First order shear deformation theory ($\kappa^2 = 5/6$).
§ Euler–Bernoulli and Timoshenko ($\kappa^2 = 5/6$) beam theories.
¶ $M = 5$: Present solution.

laminated composite beams has been studied by the Euler–Bernoulli and Timoshenko beam theories in reference [14]. A good agreement is obtained with the references for the symmetric cases.

6.3. NATURAL FREQUENCY AND BUCKLING STRESS OF LAMINATED COMPOSITE BEAMS

The buckling stresses can be calculated usually through stability equation (28) as eigenvalue problems. Another method to obtain the critical buckling stresses of laminated composite beams subjected to axial stress is to compute natural frequencies by increasing the absolute value of compressive stresses till the corresponding natural frequency vanishes.

In the case of a simply supported beam subjected to axial stress Λ, the natural frequency Ω_a can be expressed explicitly with reference to the natural frequency Ω_0 of a beam without

TABLE 5

Natural frequencies and buckling stresses of cross-ply laminated beams ($M = 5$, $r = 1$, $[0°/90°/0°/90°/\cdots]$, $K = 2$–11 *and* 100–101; *Material* 4)

No. of layers	$L/H = 2$		$L/H = 5$		$L/H = 10$	
	Ω	Λ	Ω	Λ	Ω	Λ
2	0·7423	1·0861	0·1976	3·0062	0·05772	4·1046
4	0·7355	1·0664	0·2269	3·9626	0·07791	7·4772
6	0·7559	1·1262	0·2428	4·5375	0·08302	8·4901
8	0·7910	1·2332	0·2544	4·9837	0·08567	9·0408
10	0·8010	1·2646	0·2572	5·0951	0·08644	9·2045
100	0·8251	1·3418	0·2622	5·2915	0·08774	9·4840
3	0·8495	1·4226	0·2785	5·9721	0·1041	13·3483
5	0·8173	1·3167	0·2732	5·7471	0·09969	12·2423
7	0·8340	1·3710	0·2781	5·9544	0·09837	11·9213
9	0·8456	1·4094	0·2798	6·0264	0·09715	11·6275
11	0·8482	1·4180	0·2781	5·9548	0·09577	11·2992
101	0·8299	1·3577	0·2644	5·3812	0·08873	9·6995

axial stress. The relation between Ω_a and Ω_0 can be obtained from a comparison of the equations of motion as follows:

$$\Omega_a^2 = \Omega_0^2 + \frac{r^2\pi^4}{12}\left(\frac{H}{L}\right)^4 \Lambda. \tag{38}$$

When the natural frequency Ω_a vanishes under the axial stress, elastic buckling occurs and the critical buckling stress Λ_{cr} relates to the natural frequency Ω_0 as

$$\Lambda_{cr} = -\frac{12}{r^2\pi^4}\left(\frac{L}{H}\right)^4 \Omega_0^2. \tag{39}$$

The critical buckling stresses of simply supported laminated composite beams subjected to axial stress can be predicted from the natural frequency of the beams without axial stress.

In Table 5, the natural frequency and the critical buckling stress of simply supported cross-ply laminated composite beams are shown for symmetric and skew-symmetric laminates of $L/H = 2$, 5, 10. It can be seen that relation (39) is established between the critical buckling stress and natural frequency in Table 5.

6.4. MODAL DISPLACEMENT AND STRESS DISTRIBUTIONS IN LAMINATED COMPOSITE BEAMS

Figure 2 gives an indication of the accuracy of the modal displacements and modal stresses associated with the fundamental frequencies for cross-ply laminated composite beams made of Material 4. Each of the response quantities in Figure 2 is divided by its maximum absolute value. The stress boundary conditions at the top and bottom surfaces of the beam and the continuity conditions at the interfaces between layers are satisfied accurately.

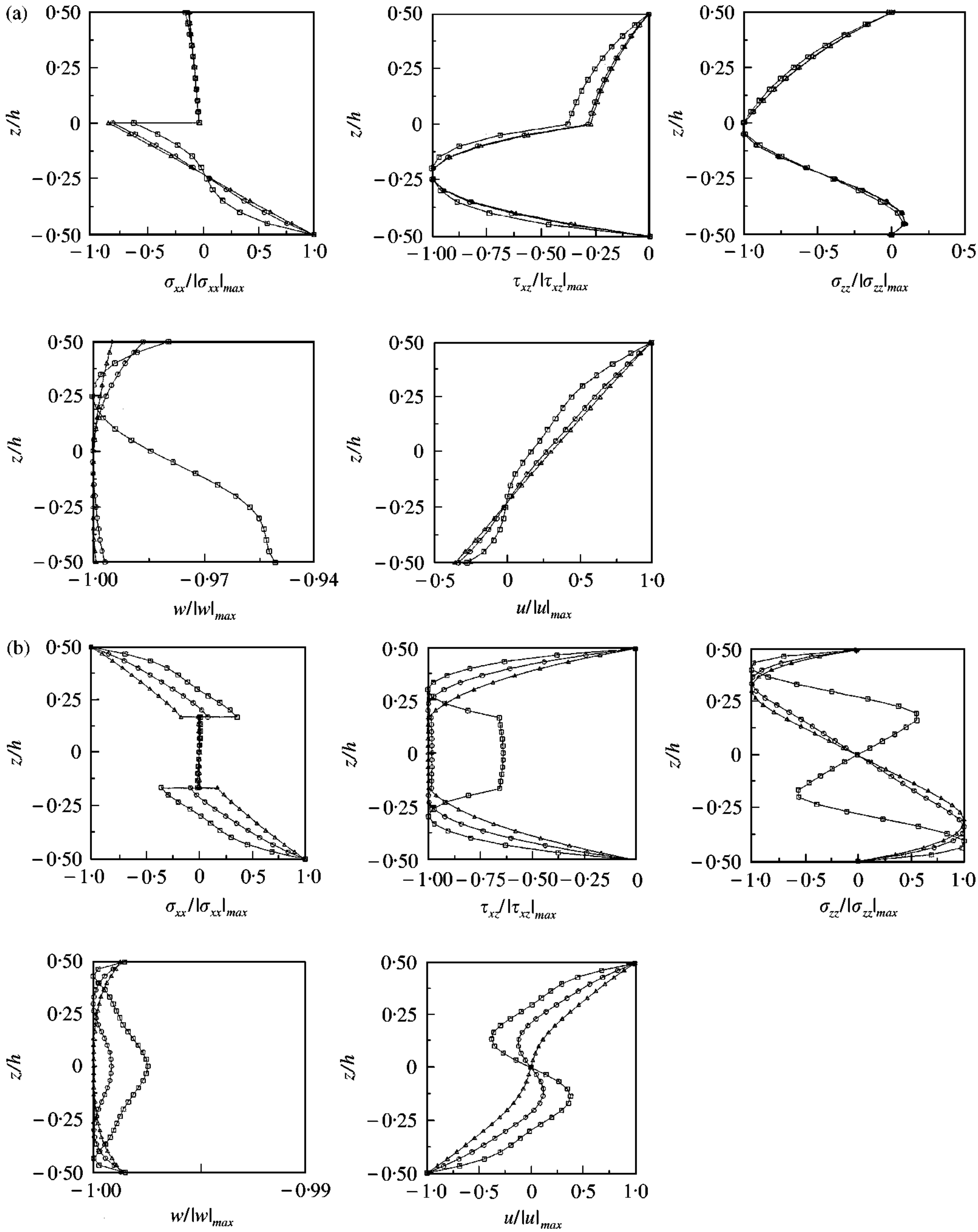

Figure 2. Displacements and stresses of cross-ply laminated composite beams. (a) [2-layers, Material 4]: —□—, $L/H = 2$; —○—, $L/H = 5$; —△—, $L/H = 10$. (b) [3-layers, Material 4]: —□—, $L/H = 2$; —○—, $L/H = 5$; —△—, $L/H = 10$. The superscript of natural frequency is longitudinal vibration mode number, r.

Since the total number of unknowns of the present global higher order theory does not increase as the number of layers increases, multilayered composite beams with a large number of layers can be analyzed without difficulty. Figure 3 shows the modal displacements and modal stresses associated with the fundamental frequency for

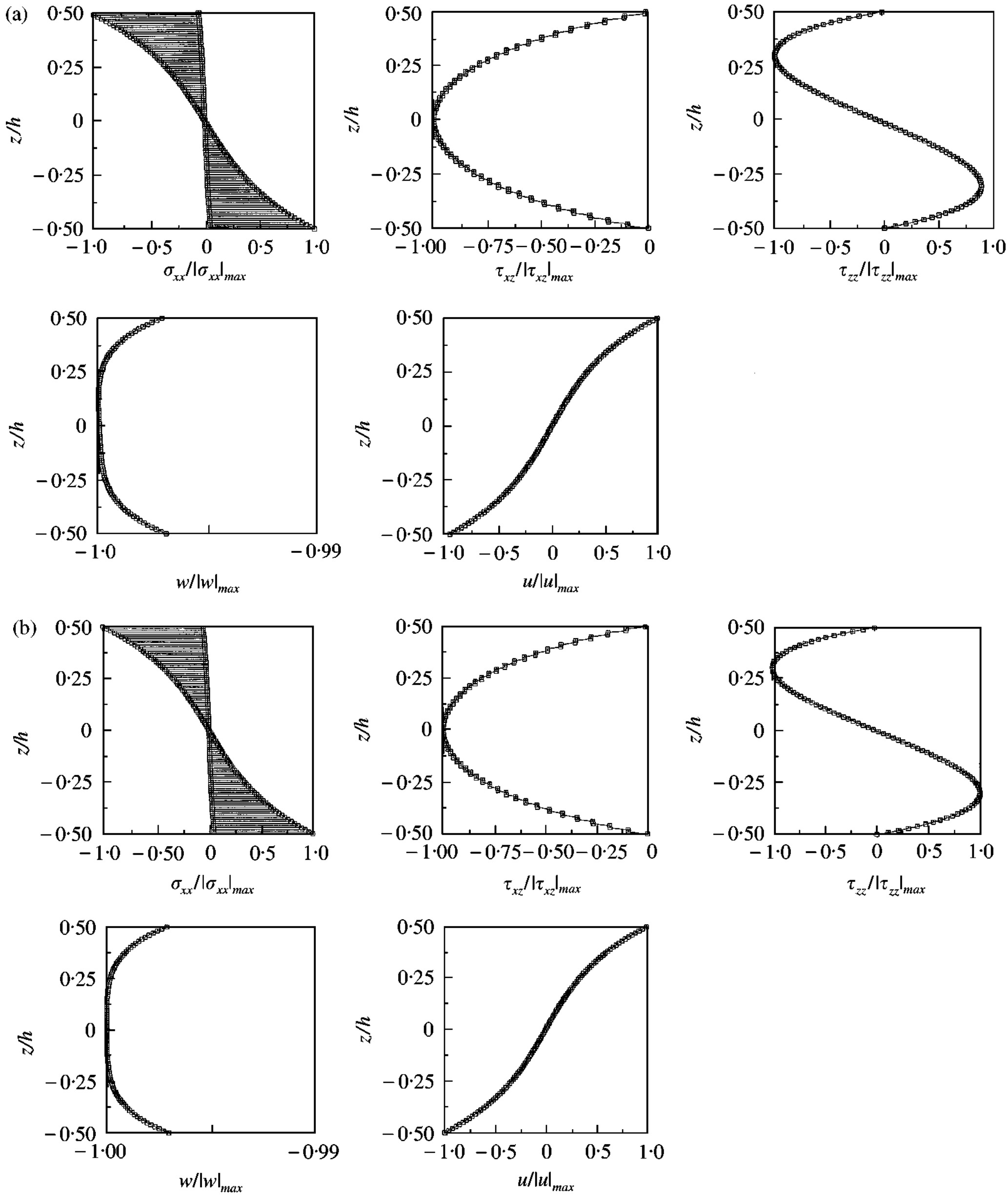

Figure 3. Displacements and stresses of cross-ply laminated composite beam. (a) [100-layers, Material 4]: —□—, $L/H = 5$. (b) [101-layers, Material 4]: —□—, $L/H = 5$. The superscript of natural frequency is longitudinal vibration mode number, r.

multilayered composite beams made of Material 4. The corresponding natural frequencies to the modal displacement and stress components in Figures 2 and 3 are shown in Table 5.

7. CONCLUSIONS

Natural frequencies and buckling stresses of simply supported multilayered composite beams have been analyzed by using a global higher order beam theory. In order to analyze the complete effects of higher order deformations on the natural frequencies and buckling stresses of cross-ply laminated composite beams, various orders of the expanded approximate laminate theories have been presented. It is shown through the numerical examples that the present global higher order theories can provide accurate results for natural frequencies and buckling stresses of general cross-ply laminated composite beams. The total number of unknowns is not dependent on the number of layers in any multilayered beams. It should be pointed out that the present theory has the advantage of predicting natural frequencies of multilayered composite beams without increasing the unknowns involved as the number of layers increases.

The distribution of modal displacements and modal stresses in the depth direction has also been obtained accurately in the ply level. The modal transverse shear and normal stresses have been obtained by integrating the three-dimensional equations of motion in the depth direction. The stress boundary conditions at the top and bottom surfaces of the beam and the continuity conditions at the interfaces between layers have been satisfied.

It has been shown that a global higher order plate theory can predict not only the natural frequency and buckling stress but also the accurate distribution of modal displacements and stress components in multilayered composite beams.

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